VARIATIONAL PRINCIPLES OF THE NONLINEAR THEORY OF ELASTICITY

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Eight variational principles of the classical theory of elasticity [1] are obtained in a different form for the case of finite deformations of an elastic body. The deviation from the calssical theory consists of the fact that the dual tensors are not symmetrical. These tensors are represented here by the Piola stress tensor and the gradient tensor of the radius vector of a point of the deformed body.

Volume and surface equations of equilibrium of an elastic body can be written as [2, 3]

$$\nabla \cdot \mathrm{D} + \rho_0 \mathrm{K} = 0 \quad \text{in} \quad v \tag{1}$$

$$\mathbf{n} \cdot \mathrm{D} = \mathbf{F}^\circ \quad \text{on} \quad o \tag{2}$$

Here ∇ is the Del operator in the metric of the undeformed state, D is the nonsymmetric Piola stress tensor, ρ_0 is the density of the undeformed material, K is the body force vector, $\mathbf{F}^\circ$ is the surface force vector per unit area of the undeformed body, v is the volume occupied by the undeformed body, o is the surface bounding v, and $\mathbf{n}$ is a normal to the surface o.

The general solution of (1) has the form

$$\mathrm{D} = \nabla \times \Phi + U\mathrm{E} \qquad (\rho_0 \mathrm{K} = -\nabla U) \tag{3}$$

where Φ is an arbitrary, twice-differentiable tensor, U is the potential of the volume forces and E is the unit tensor. The tensor D is a potential tensor function of the gradient tensor of the radius vector of a point of the deformed body [4] and is given by

$$\mathrm{D} = \frac{dW}{d\mathrm{C}}, \qquad \mathrm{C} = \nabla \mathrm{R} \tag{4}$$

where W is the specific potential strain energy.

We further bring into consideration the specific complementary strain work as a function of the components of the Piola stress tensor, related to W through the Legendre transformation

$$B = \mathrm{D} \cdot\cdot \mathrm{C}^T - W \tag{5}$$

The property of the Legendre transformation implies that

$$\mathrm{C} = \frac{dB(\mathrm{D})}{d\mathrm{D}} \tag{6}$$

A method of expressing the specific complementary strain work in terms of the Piola stress tensor is given in [4] for an isotropic body. The tensor C defined by (6) is not, in general, a gradient of a vector. The following condition of compatibility represents the necessary and sufficient condition for C to be a gradient:

$$\nabla \times \mathrm{C} = 0 \tag{7}$$

Considerations that follow are based on the following identities valid for any differentiable tensors P and Q and for the vector $\mathbf{a}$:

$$\nabla\cdot(\mathbf{P}\cdot\mathbf{a}) = \mathbf{a}\cdot(\nabla\cdot\mathbf{P}) + \mathbf{P}^T\cdot\cdot\nabla\mathbf{a} \tag{8}$$

$$I_1[\nabla\times(\mathbf{P}\cdot\mathbf{Q})] = -\nabla\cdot\boldsymbol{\epsilon}\cdot\cdot(\mathbf{P}\cdot\mathbf{Q}) = \mathbf{Q}\cdot\cdot(\nabla\times\mathbf{P}) - \mathbf{P}^T\cdot\cdot(\nabla\times\mathbf{Q}^T) \tag{9}$$

Here I_1 denotes the first tensor invariant and $\epsilon = -\mathrm{E}\times\mathrm{E}$ is an isotropic tensor of the third rank (the Levi-Civita tensor). From (8) and (9) we obtain the following integral identities

$$\iiint_v \mathbf{a}\cdot(\nabla\cdot P)\,d\tau = -\iiint_v \mathbf{P}^T\cdot\cdot\nabla\mathbf{a}\,d\tau + \iint_o \mathbf{n}\cdot\mathbf{P}\cdot\mathbf{a}\,do \tag{10}$$

$$\iiint_v \mathbf{Q}\cdot\cdot(\nabla\times\mathbf{P})\,d\tau = \iiint_v \mathbf{P}^T\cdot\cdot(\nabla\times\mathbf{Q}^T)\,d\tau - \iint_o \mathbf{n}\cdot\epsilon\cdot\cdot(\mathbf{P}\cdot\mathbf{Q})\,do \tag{11}$$

Let o_1 be the part of the surface on which the external forces are given and let the displacements $\mathbf{R} = \mathbf{R}^*$ be given on $o_2 = o - o_1$. The external forces are assumed to be the "dead type" loads, i.e. the vectors $\mathbf{K}$ and $\mathbf{F}^\circ$ do not depend on the displacements.

First principle. Consider the functional

$$J_1(\mathbf{R}) = \iiint_v [W(\mathbf{R}) - \rho_0\mathbf{K}\cdot\mathbf{u}]\,d\tau - \iint_{o_1} \mathbf{F}^\circ\cdot\mathbf{u}\,do \tag{12}$$

over the displacement vector $\mathbf{u}$. The conditions that $\delta J_1(\mathrm{R}) = 0$ and $\delta\mathbf{R} = 0$ on o_2 are equivalent to the following equations in terms of displacements

$$\nabla\cdot\mathrm{D}(\mathbf{R}) + \rho_0\mathbf{K} = 0 \quad \text{in} \quad v$$

and the boundary conditions

$$\mathbf{n}\cdot\mathrm{D}(\mathbf{R}) = \mathbf{F}^\circ \quad \text{on} \quad o_1$$

Proof. By (4) and (10) we have

$$\delta J_1 = \iiint_v (\mathrm{D}\cdot\cdot\delta\nabla\mathbf{R}^T - \rho_0\mathbf{K}\cdot\delta\mathbf{R})\,d\tau - \iint_{o_1}\mathbf{F}^\circ\cdot\delta\mathbf{R}do =$$
$$= \iint_{o_1}(\mathbf{n}\cdot\mathrm{D} - \mathbf{F}^\circ)\cdot\delta\mathbf{R}do - \iiint_v \delta\mathbf{R}\cdot(\nabla\cdot\mathrm{D} + \rho_0\mathbf{K})\,d\tau$$

Second principle. We consider a functional over the displacement vector and the Piola stress tensor

$$J_2(\mathbf{R},\mathrm{D}) = \iiint_v [\mathrm{D}\cdot\cdot\nabla\mathbf{R}^T - B(\mathrm{D}) - \rho_0\mathbf{K}\cdot\mathbf{u}]\,d\tau -$$
$$- \iint_{o_1}\mathbf{F}^\circ\cdot\mathbf{u}\,do - \iint_{o_2}\mathbf{n}\cdot\mathrm{D}\cdot(\mathbf{R} - \mathbf{R}^*)\,do$$

The condition $\delta J_2(\mathbf{R},\mathrm{D}) = 0$ is equivalent to the equations

$$\nabla\cdot\mathrm{D} + \rho_0\mathbf{K} = 0, \quad \nabla\mathbf{R} = \frac{dB(\mathrm{D})}{d\mathrm{D}} \quad \text{in } v$$

and the boundary conditions

$$\mathbf{n}\cdot\mathrm{D} = \mathbf{F}^\circ \quad \text{on } o_1, \qquad \mathbf{R} = \mathbf{R}^* \quad \text{on } o_2$$

Proof. By (6) and (10) we have

$$\delta J_2 = -\iiint_v (\nabla\cdot\mathrm{D} + \rho_0\mathbf{K})\cdot\delta\mathbf{R}\,d\tau + \iiint_v\left(\nabla\mathbf{R} - \frac{dB}{d\mathrm{D}}\right)\cdot\cdot\delta\mathrm{D}^T\,d\tau +$$

$$+\iint_{o_1}(\mathbf{n}\cdot\mathrm{D}-\mathbf{F}^\circ)\cdot\delta\mathbf{R}\,do-\iint_{o_2}\mathbf{n}\cdot\delta\mathrm{D}\cdot(\mathbf{R}-\mathbf{R}^*)\,do$$

Third principle. The functional

$$J_3(\mathbf{R},\mathrm{D},\mathrm{C})=\iiint_v[W(\mathrm{C})-\mathrm{D}^T\cdot\cdot(\mathrm{C}-\nabla\mathbf{R})-\rho_0\mathbf{K}\cdot\mathbf{u}]\,d\tau-$$
$$-\iint_{o_1}\mathbf{F}^\circ\cdot\mathbf{u}\,do-\iint_{o_2}\mathbf{n}\cdot\mathrm{D}\cdot(\mathbf{R}-\mathbf{R}^*)\,do$$

The condition $\delta J_3(\mathrm{R},\ \mathrm{D},\ \mathrm{C})=0$ is equivalent to

$$\nabla\cdot\mathrm{D}+\rho_0\mathbf{K}=0,\quad \mathrm{D}=\frac{dW}{d\mathrm{C}},\quad \mathrm{C}=\nabla\mathbf{R}\quad\text{in } v$$

and the boundary conditions

$$\mathbf{n}\cdot\mathrm{D}=\mathbf{F}^\circ\quad\text{on } o_1,\qquad \mathbf{R}=\mathbf{R}^*\quad\text{on } o_2$$

Proof.

$$\delta J_3=\iiint_v\left(\frac{dW}{d\mathrm{C}}-\mathrm{D}\right)\cdot\cdot\delta\mathrm{C}^T\,d\tau-\iiint_v(\nabla\cdot\mathrm{D}+\rho_0\mathbf{K})\cdot\delta\mathbf{R}\,d\tau-$$
$$-\iiint_v(\mathrm{C}-\nabla\mathbf{R})\cdot\cdot\delta\mathrm{D}^T\,d\tau+\iint_{o_1}(\mathbf{n}\cdot\mathrm{D}-\mathbf{F}^\circ)\cdot\delta\mathbf{R}\,do-\iint_{o_2}\mathbf{n}\cdot\delta\mathrm{D}\cdot(\mathbf{R}-\mathbf{R}^*)\,do$$

Fourth principle. We assume that $o=o_1$ and denote by $B\,[\mathrm{D}\,(\Phi)]$ the specific complementary strain work expressed, according to (3), by the tensor Φ. The functional is

$$J_4(\Phi)=\iiint_v B\,[\mathrm{D}\,(\Phi)]\,d\tau$$

The conditions $\delta J_4=0$ and $\delta\Phi=0$ on $o=o_1$ are equivalent to the following compatibility equations for the tensor Φ:

$$\nabla\times\mathrm{C}\,(\Phi)=0\quad\text{in } v$$

Proof. By (11) and (6) we have

$$\delta J_4=\iiint_v\frac{dB}{d\mathrm{D}}\cdot\cdot(\nabla\times\delta\Phi)^T\,d\tau=\iiint_v\mathrm{C}^T\cdot\cdot(\nabla\times\delta\Phi)\,d\tau=$$
$$=\iiint_v\delta\Phi^T\cdot\cdot(\nabla\times\mathrm{C})\,d\tau-\iint_o\mathbf{n}\cdot\epsilon\cdot\cdot(\delta\Phi\cdot\mathrm{C}^T)\,do$$

The variational principle giving rise to the compatibility equations written in terms of the components of the tensor D, was given in [4] in a slightly different form.

Fifth principle. The functional

$$J_5(\mathrm{C},\Phi)=\iiint_v[\mathrm{D}\,(\Phi)\cdot\cdot\mathrm{C}^T-W(\mathrm{C})]\,d\tau$$

The conditions $\delta J_5=0$ and $\delta\Phi=0$ on $o=o_1$ are equivalent to the equations

$$\mathrm{D}\,(\Phi)=\frac{dW}{d\mathrm{C}},\quad \nabla\times\mathrm{C}=0\quad\text{in } v$$

Proof.

$$\delta J_5=\iiint_v\left(\mathrm{D}-\frac{dW}{d\mathrm{C}}\right)\cdot\cdot\delta\mathrm{C}^T\,d\tau+\iiint_v\delta\Phi^T\cdot\cdot(\nabla\times\mathrm{C})\,d\tau-\iint_o\mathbf{n}\cdot\epsilon\cdot\cdot(\delta\Phi\cdot\mathrm{C}^T)\,do$$

Sixth principle. The functional

$$J_6(\mathrm{C}, \mathrm{D}, \Phi) = \iiint\limits_v [B(\mathrm{D}) - \mathrm{C}^T\cdot\cdot(\mathrm{D} - \nabla\times\Phi - U\mathrm{E})]\, d\tau$$

The conditions $\delta J_6 = 0$ and $\delta\Phi = 0$ on $o = o_1$ are equivalent to the equations

$$\mathrm{C} = \frac{dB}{dD}, \quad \nabla\times\mathrm{C} = 0, \qquad \mathrm{D} = \nabla\times\Phi + U\mathrm{E}$$

Proof.

$$\delta J_6 = \iiint\limits_v \left(\frac{dB}{d\mathrm{D}} - \mathrm{C}\right)\cdot\cdot\,\delta\mathrm{D}^T d\tau - \iiint\limits_v (\mathrm{D} - \nabla\times\Phi - U\mathrm{E})\cdot\cdot\,\delta\mathrm{C}^T d\tau +$$

$$+ \iiint\limits_v \delta\Phi^T\cdot\cdot(\nabla\times\mathrm{C})\, d\tau - \iint\limits_o \mathbf{n}\cdot\epsilon\cdot\cdot(\delta\Phi\cdot\mathrm{C}^T)\, do$$

Seventh principle. The functional

$$J_7(\mathbf{R}, \mathrm{D}, \mathrm{C}, \Phi) = \iiint\limits_v [\mathrm{D}\cdot\cdot\nabla\mathbf{R}^T - B(\mathrm{D}) + \mathrm{D}(\Phi)\cdot\cdot\mathrm{C}^T - W(\mathrm{C})]\, d\tau -$$

$$- \iiint\limits_v \rho_0\mathbf{K}\cdot\mathbf{u}\, d\tau - \iint\limits_o \mathbf{F}^\circ\cdot\mathbf{u}\, do$$

The conditions $\delta J_7 = 0$ and $\delta\Phi = 0$ on $o = o_1$ are equivalent to the equations

$$\nabla\cdot\mathrm{D} + \rho_0\mathbf{K} = 0, \quad \mathrm{D}(\Phi) = \frac{dW}{d\mathrm{C}}, \quad \nabla\mathbf{R} = \frac{dB}{d\mathrm{D}}, \quad \nabla\times\mathrm{C} = 0$$

and the boundary conditions

$$\mathbf{n}\cdot\mathrm{D} = \mathbf{F}^\circ \quad \text{on } o = o_1$$

Proof.

$$\delta J_7 = -\iiint\limits_v (\nabla\cdot\mathrm{D} + \rho_0\mathbf{K})\cdot\delta\mathbf{R}\, d\tau + \iiint\limits_v \left(\nabla\mathbf{R} - \frac{dB}{d\mathrm{D}}\right)\cdot\cdot\,\delta\mathrm{D}^T d\tau +$$

$$+ \iiint\limits_v \left(\mathrm{D} - \frac{dW}{d\mathrm{C}}\right)\cdot\cdot\,\delta\mathrm{C}^T d\tau + \iint\limits_o (\mathbf{n}\cdot\mathrm{D} - \mathbf{F}^\circ)\cdot\delta\mathbf{R}\, do -$$

$$- \iint\limits_o \mathbf{n}\cdot\epsilon\cdot\cdot(\delta\Phi\cdot\mathrm{C}^T)\, do + \iiint\limits_v \delta\Phi^T\cdot\cdot(\nabla\times\mathrm{C})\, d\tau$$

Eighth principle. The compatibility equations for the tensor C follow from the fact that the functional $J_8(\mathrm{C})$ is stationary when $\delta\mathrm{C} = 0$ on o

$$J_8(\mathrm{C}) = \frac{1}{2}\iiint\limits_v \mathrm{C}^T\cdot\cdot(\nabla\times\mathrm{C})\, d\tau$$

Proof. By (11) we have

$$\delta J_8 = \frac{1}{2}\iiint\limits_v [\delta\mathrm{C}^T\cdot\cdot(\nabla\times\mathrm{C}) + \mathrm{C}^T\cdot\cdot(\nabla\times\delta\mathrm{C})]\, d\tau = \frac{1}{2}\iiint\limits_v \delta\mathrm{C}^T\cdot\cdot(\nabla\times\mathrm{C})\, d\tau +$$

$$+ \frac{1}{2}\iiint\limits_v \delta\mathrm{C}^T\cdot\cdot(\nabla\times\mathrm{C})\, d\tau - \frac{1}{2}\iint\limits_o \mathbf{n}\cdot\epsilon\cdot\cdot(\delta\mathrm{C}\cdot\mathrm{C}^T)\, do = \iiint\limits_v \delta\mathrm{C}^T\cdot\cdot(\nabla\times\mathrm{C})\, d\tau$$

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ON THE STATISTICAL THEORY OF VISCOELASTIC PROPERTIES OF ASYMMETRIC MEDIA

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Within the scope of the Kubo linear reaction based on the classical Gibbs formalism without involving known additional representations, expressions in terms of the time correlation functions are obtained for four tensors of the viscosity coefficients of an asymmetric medium. Independently of the time correlation function apparatus, expressions are established for the ultimate high-frequency and adiabatic elastic moduli by analyzing the increments of the stress tensors upon application of a small strain.

Macroscopic phenomena of the internal (rotational) degrees of freedom are the center of attention of phenomenological theories of asymmetric media (see [1−4], for example). According to these latter, the motion of a continuum is described by the field of mean angular velocities of the natural particle rotation as well as by the field of mean translational velocities. The state of strain is defined by two strain rate tensors (two strain tensors), and the state of stress by tensors of the ordinary and couple stresses.

Moreover, many important characteristics of the behavior of asymmetric media cannot be determined within the scope of the phenomenological approach. An experimental study also encounters a number of difficulties.

Modern methods of the statistical theory of irreversible processes provide the possibility, in principle, of a theoretical determination of the characteristics of the behavior of the systems under consideration.

Earlier, conservation laws for asymmetric media [5] were given a statistical foundation on the basis of the Liouville equation. Conservation laws and irreversible processes in these media have been examined in [6] by the method of a nonequilibrium statistical operator. The method of correlative functions of conditional distributions [7, 8] has been applied in giving a statistical foundation to the conservation laws and singularities of the kinematics of a medium. The expressions obtained for the stress tensors and the couple stresses afforded a possi-